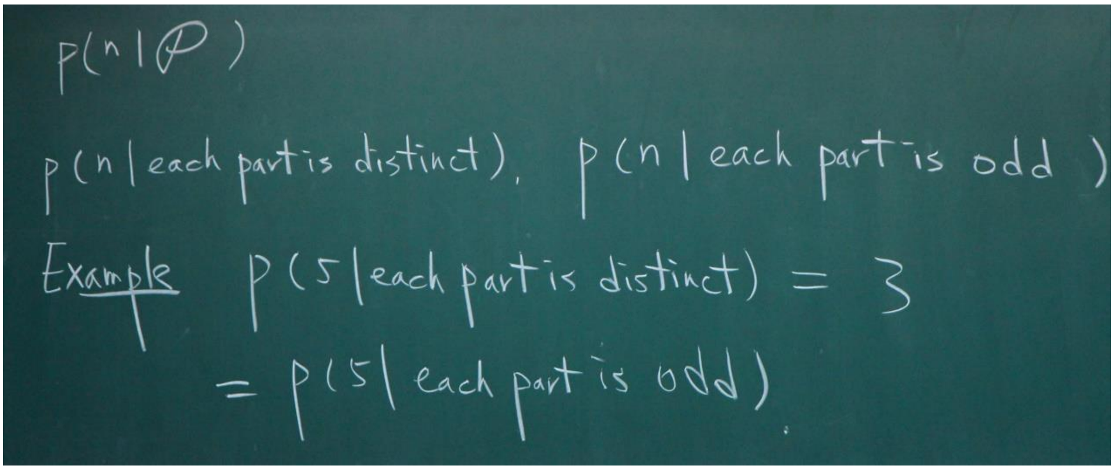
【10920 趙啟超教授離散數學 / 第 17 堂版書】



$p(n \mid \text{each part is distinct})$, $p(n \mid \text{each part is odd})$

Example $p(5 \mid \text{each part is distinct}) = 3$
$= p(5 \mid \text{each part is odd})$.

Example

$$\begin{aligned} 8 &= 8 \\ &= 7+1 \\ &= 6+2 \\ &= 5+3 \\ &= 5+2+1 \\ &= 4+3+1 \end{aligned}$$

$\therefore p(8 \mid \text{each part is distinct}) = 6$

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$\therefore p(8 \mid \text{each part is odd}) = 6$

$p(n \mid \mathcal{P})$

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$\therefore p(8 \mid \text{each part is odd}) = 6$

Is this a coincidence? No.

<u>Property</u> $p(n \mid \text{each part is distinct})$
$= p(n \mid \text{each part is odd})$.

<u>Proof</u> Let the generating functions for $p(n \mid \text{each part is distinct})$ and $p(n \mid \text{each part is odd})$ be $P_d(x)$ and $P_o(x)$, respectively.

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Then

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\begin{aligned}
P_d(x) &= (1+x)(1+x^2)(1+x^3)\cdots\cdots \\
&= \frac{1\cancel{-x^2}}{1-x}\cdot\frac{1\cancel{-x^4}}{1\cancel{-x^2}}\cdot\frac{1\cancel{-x^6}}{1-x^3}\cdot\cdots\cdots \\
&= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots\cdots} \\
&= (1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots \\
&= P_o(x).
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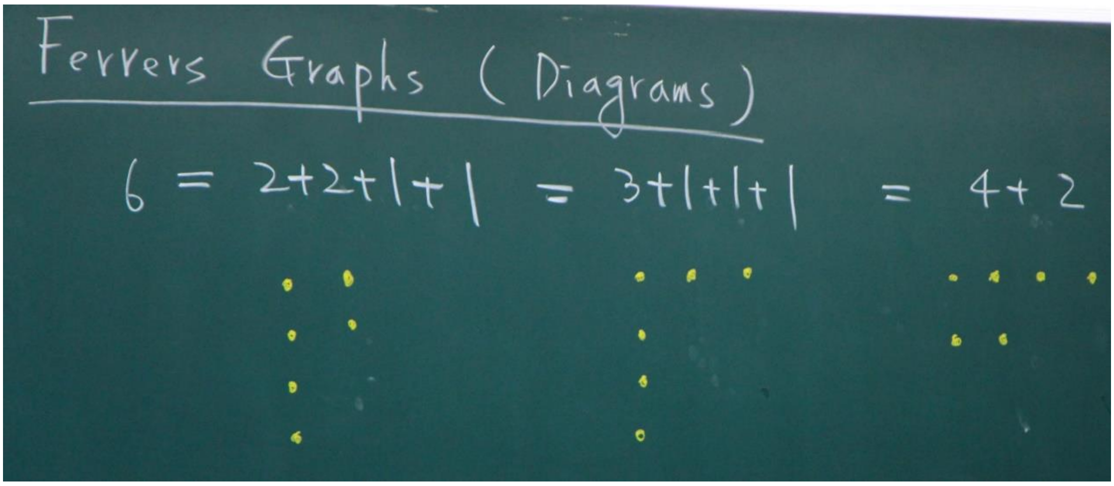
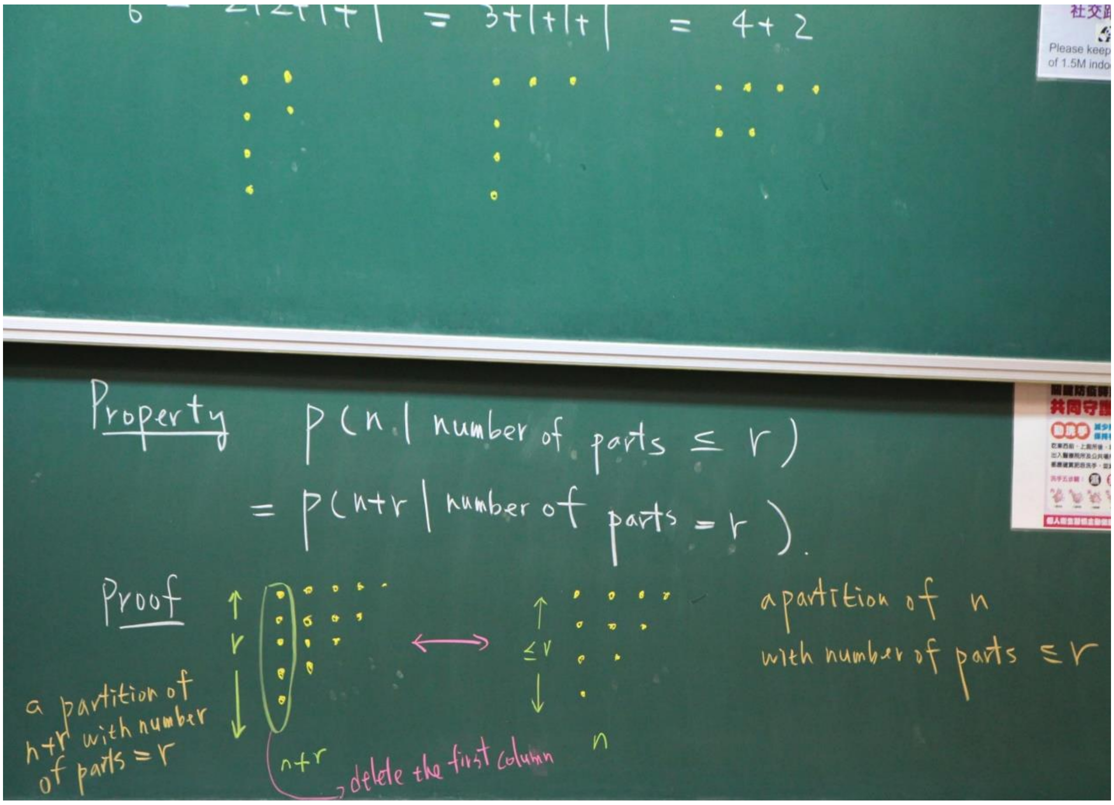
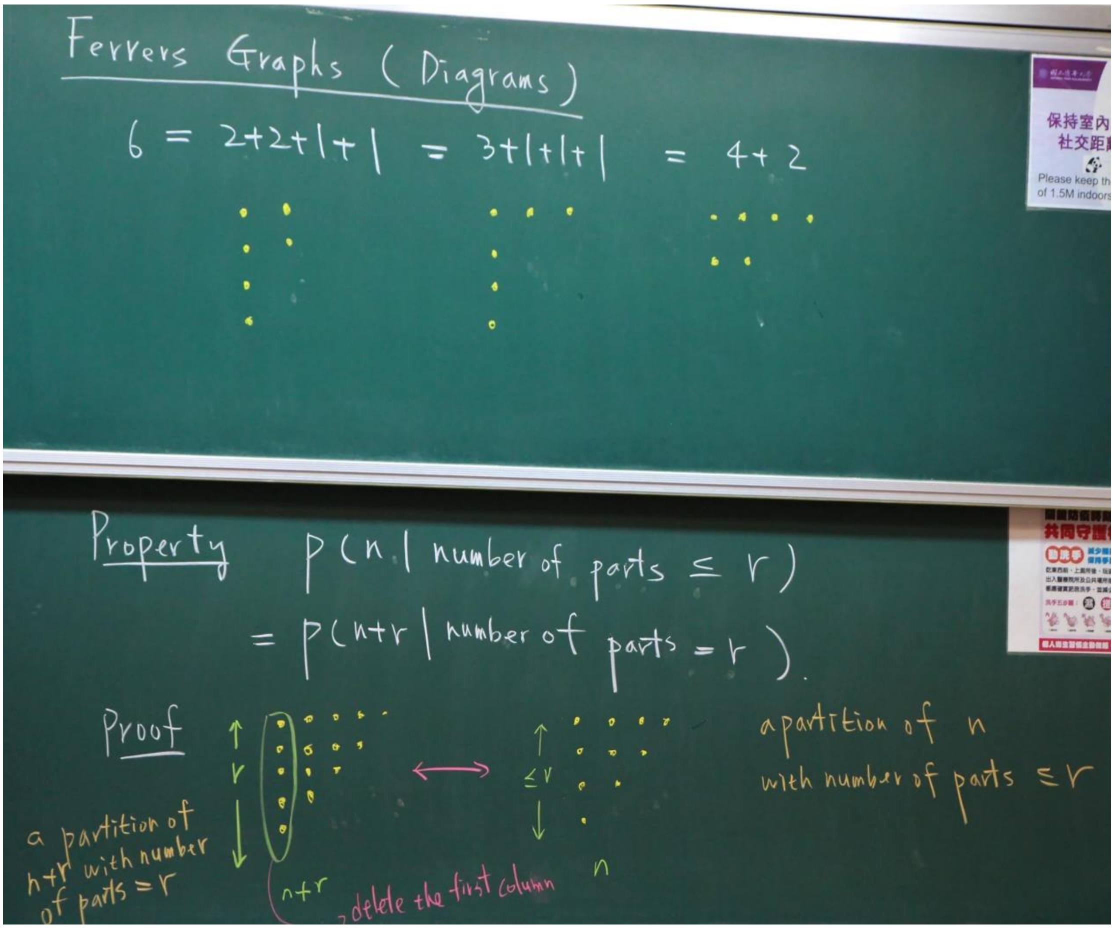

Ferrers Graphs (Diagrams)
6 = 2+2+1+1 = 3+1+1+1 = 4+2


= 3+1+1+1 = 4+2
Property p(n | number of parts ≤ r)
= p(n+r | number of parts = r).
Proof
a partition of n+r with number of parts = r
r
n+r
delete the first column
≤r
n
a partition of n with number of parts ≤ r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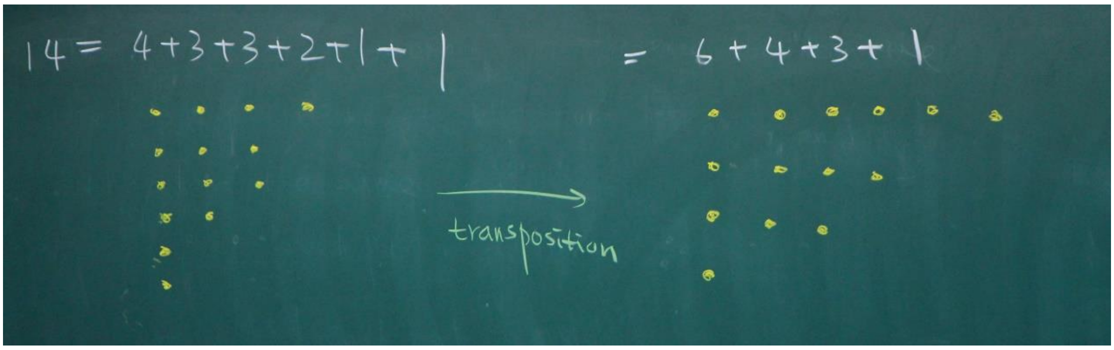
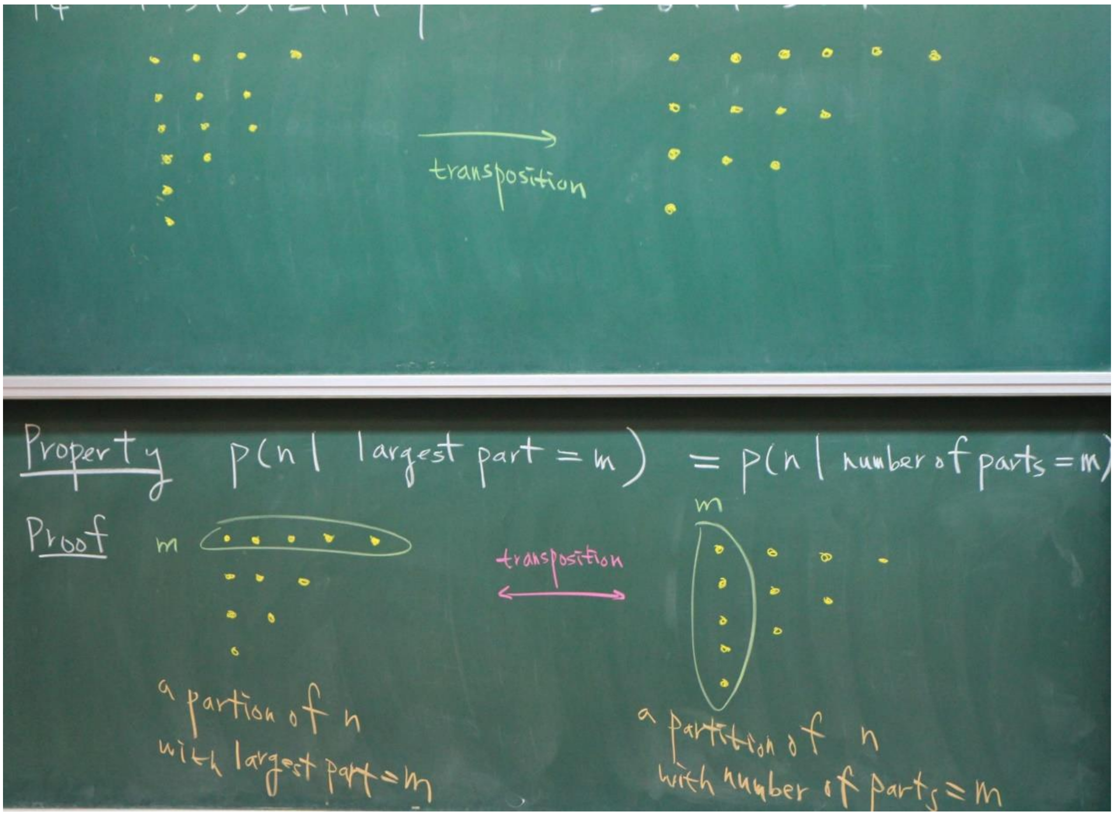

14 = 4+3+3+2+1+1 = 6+4+3+1
transposition


transposition
Property
p(n | largest part = m) = p(n | number of parts = m)
Proof
m
transposition
m
a partion of n
with largest part = m
a partition of n
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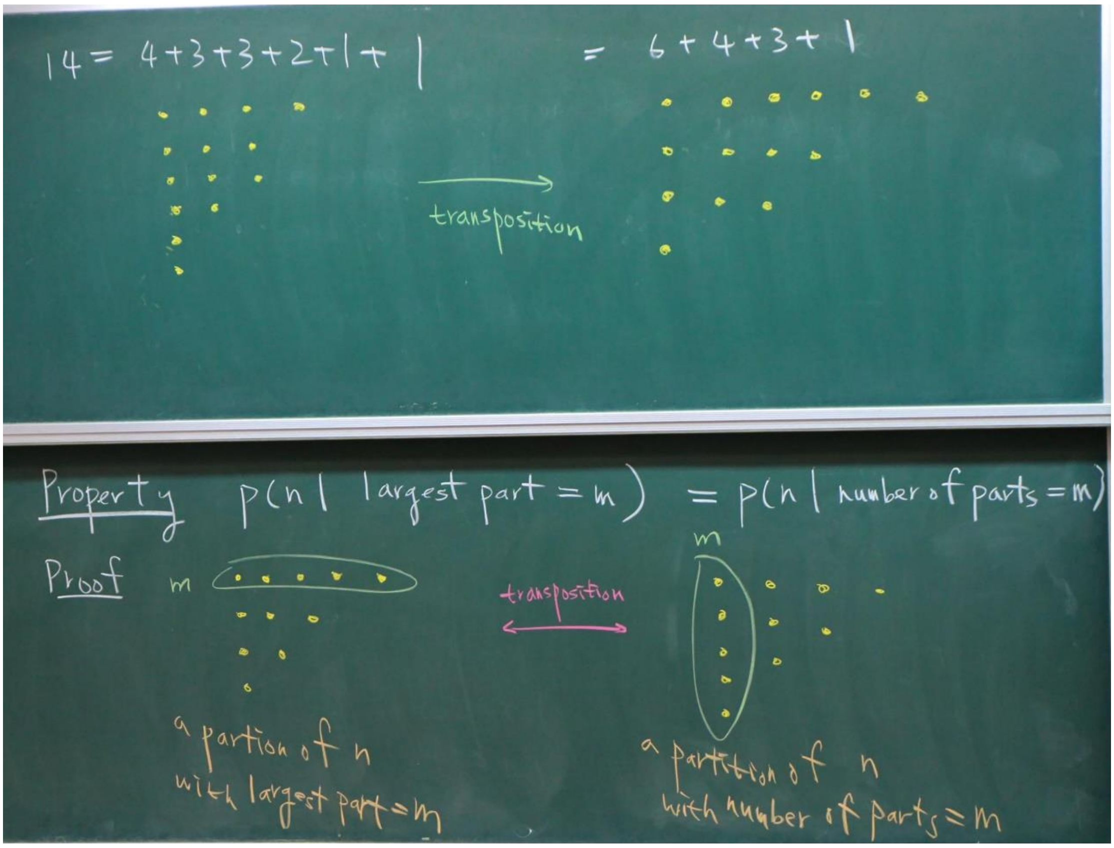
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$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

$$(1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+$$

$$= P_o(x).$$

There is a one-to-one correspondence between the sets of partitions of the two kinds, and so they have the same cardinality. ∎

Complexity of Algorithms

An algorithm is a list of precise instructions designed to solve a particular type of problem — not just one special case.

For analysis of an algorithm:

1. Describe the algorithm precisely.
2. Define the size of an instance n
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Example Assume one million operations per second

MIPs (million instructions per second)

$f(n)$	$n=20$	$n=40$	$n=60$
n	0,00002 sec	0,00004 sec	0,00006 sec
n^2	0,0004 sec	0,0016 sec	0,0036 sec
n^3	0,008 sec	0,064 sec	0,216 sec
2^n	1,0 sec	12,7 days	366 centuries

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Def Let f be a function from $\mathbb{N}$ to $\mathbb{N}$.
We say that $f(n)$ is $O(g(n))$
if there exist positive integers k and n_0
such that

$$f(n) \le k\, g(n)$$

for all $n \ge n_0$.

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Remark The symbol $O(g(n))$ is pronounced "big-oh of $g(n)$."

Example $f(n) = 3n^3 - 20n^2 + 5n - 11$

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Example $n^2 + 17n + 3$ is $O(n^2)$.

$2^n + 3n^5 + 12n^4$ is $O(2^n)$.

Example Sorting algorithms

A sequence of distinct numbers $x_1, x_2, \cdots, x_n$
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"Bubble sort"

for $j := 1$ to $n-1$

for $i := 1$ to $n-j$

if $x_i > x_{i+1}$ then interchange x_i and x_{i+1}

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At the jth pass, $n-j$ comparisons are made,

and so total # of comparisons

$$= (n-1) + (n-2) + \cdots + 2 + 1$$

$$= \frac{n(n-1)}{2}$$

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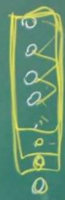

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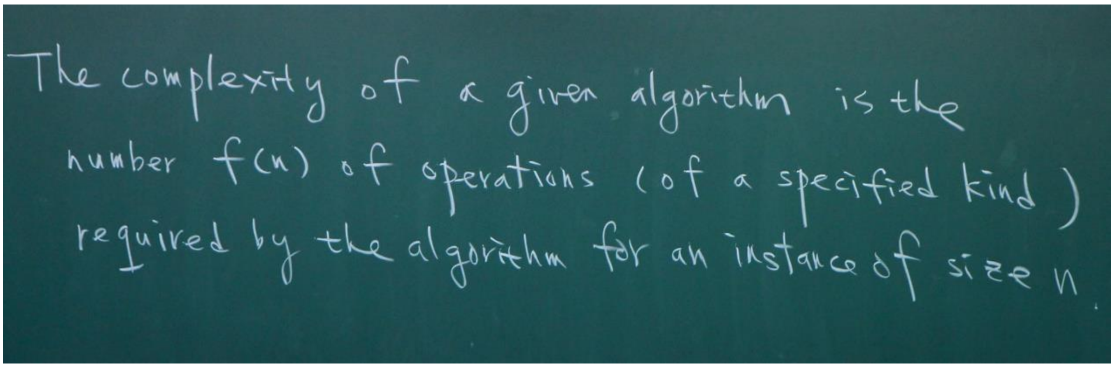
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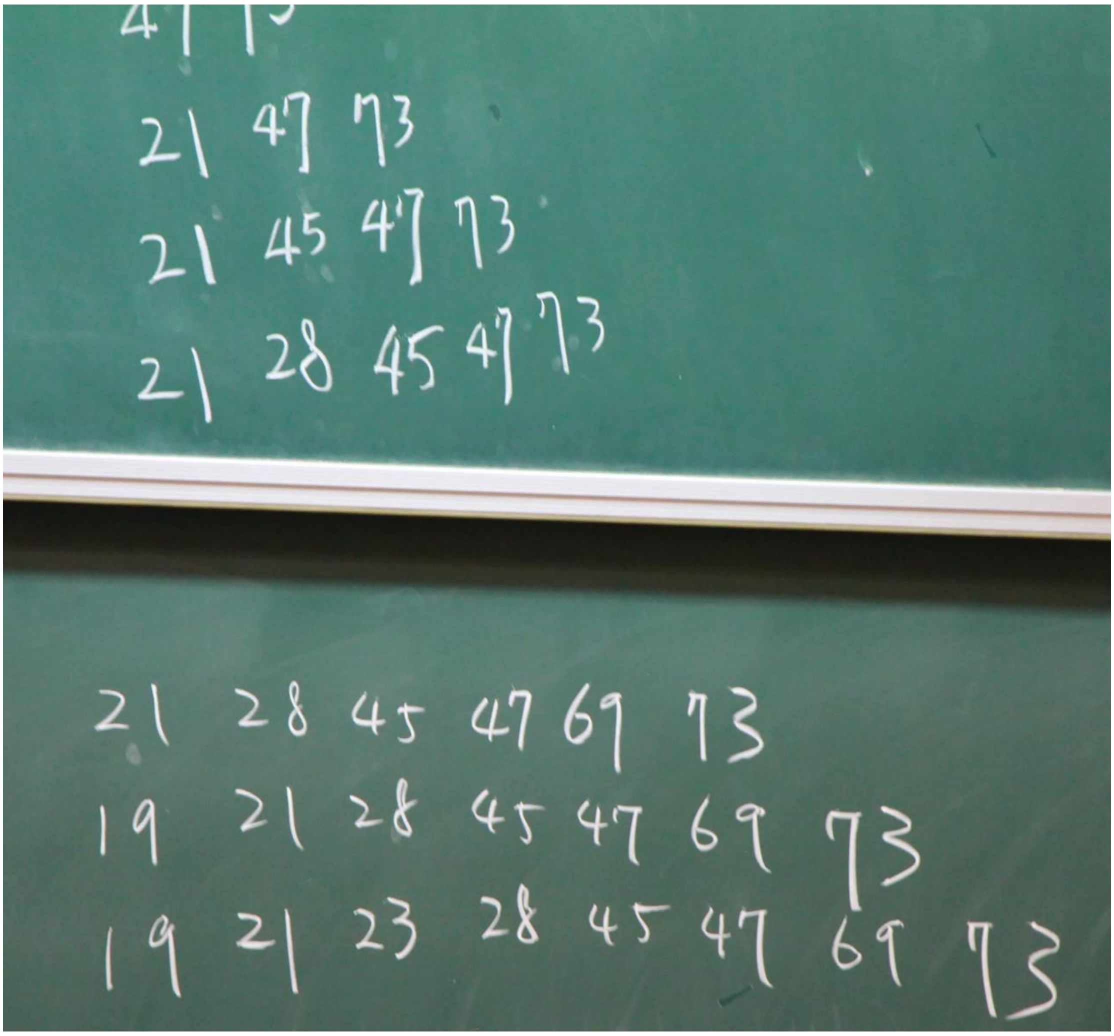
21 47 73
21 45 47 73
21 28 45 47 73
21 28 45 47 69 73
19 21 28 45 47 69 73
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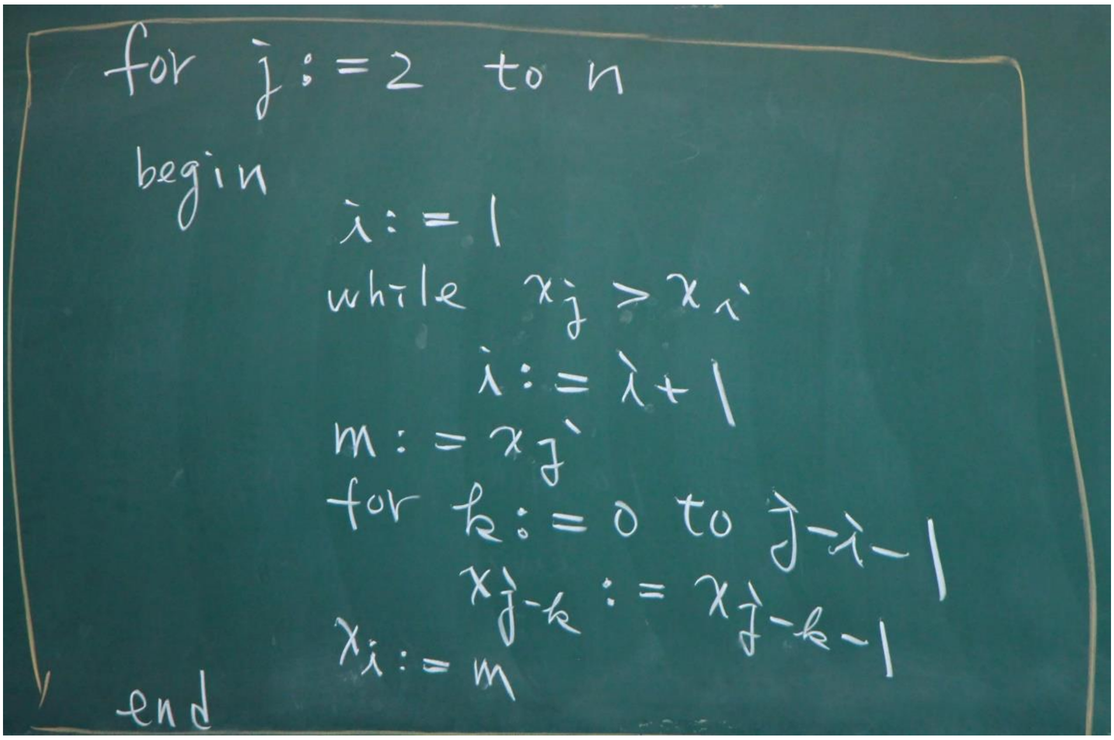
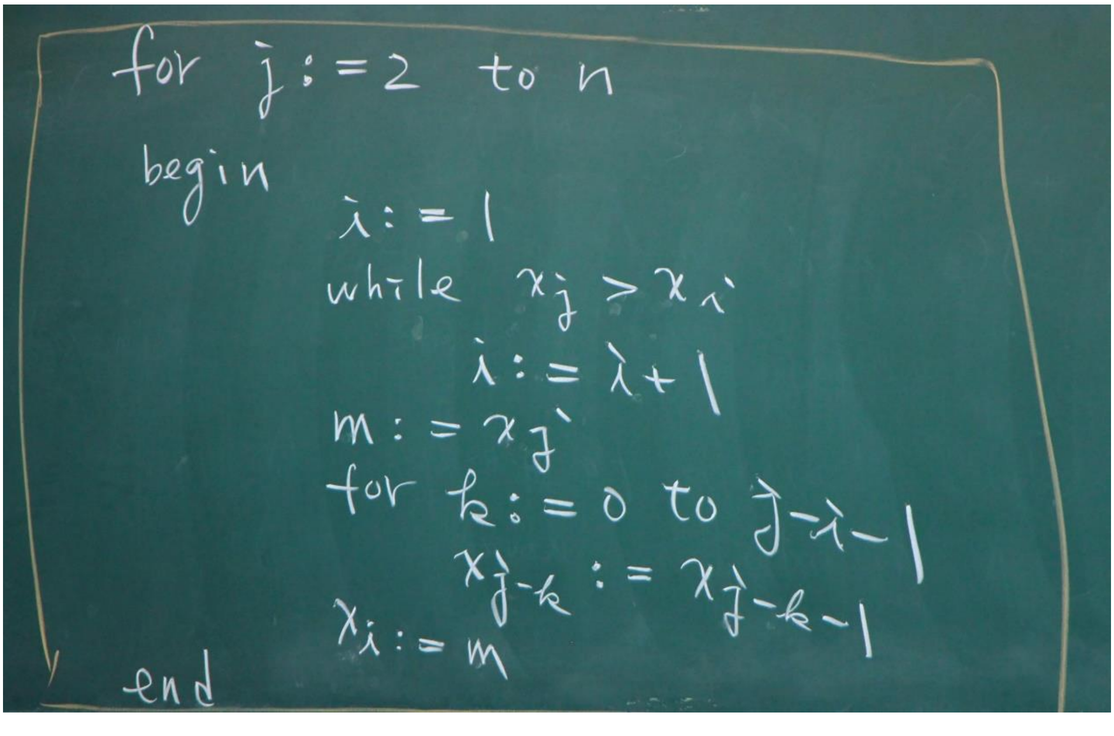

for j:=2 to n
begin
i:=1
while $x_j > x_i$
i:=i+1
m:=x_j
for k:=0 to j-i-1
$x_{j-k} := x_{j-k-1}$
x_i:=m
end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(worst-case) $= \dfrac{n(n-1)}{2}$

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There are sorting algorithms which can achieve $O(n\log n)$ comparisons (in the worst case).

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